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Mathematical Model of Migration of Spherical Particles in Tube Flow Under the Influence of Inertia and Particle-particle Interaction

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Abstract−In this paper, a mathematical model is considered of the migration of non-colloidal, spherical particles suspended in Newtonian fluid under Poiseuille flows by combining the inertial migration theory by Ho and Leal (JFM, 1974) and particle migration model in concentrated suspension by Phillips et al. (Phys. Fluids, 1992). The numerical solutions of the model equations reveal that the model set up here explains the experimental observation reported in the literature when $Re_p<1$, at least qualitatively. It was concluded that both the inertia and particle-particle interaction should be taken into account properly to understand the particle migration in tube flow of suspension regardless of particle loading.

Key words: Shear-induced Migration, Inertial Migration, Particle-particle Interaction, Suspension, Velocity Blunting, Segre-Silberberg Effect

INTRODUCTION

The flow of solid suspension and the microstructure in the suspension have been of great interest in the materials development and industrial processes such as high strength ceramics and reinforced polymer composites. Also there has been growing interest in the flow and solid distribution in the flow of blood [Cha and Beissinger, 2001], fermented broth [Lim et al., 2002], colloids [Chun and Baig, 2001] and other noble systems such as electrorheological fluids [Chin and Park, 2001], nanocomposites [Okada et al., 2002] and nanofluids [Eastman and Choi, 1996].

Segre and Silberberg [1962] first reported that a neutrally buoyant rigid particle suspended in an incompressible fluid that is undergoing a Poiseuille flow would migrate across the streamline and eventually reach an equilibrium position at 0.6 of pipe radius from the axis. Theoretically, Ho and Leal [1974] found that the migration was caused by the fluid inertia due to the presence of particles by using the method of images. Since then, these two pioneering reports on the inertial migration of particles in a tube flow researches on the cross-stream migration of particles have occupied an important position in suspension rheology. Later Leighton and Acrivos [1987] reported on particle migration due to particle-particle interaction in a concentrated suspension. Since then most studies on particle migration in semi-concentrated or concentrated suspension have been focused on the cases of vanishingly small particle Reynolds number, Re_p [MRI, Abbot et al., 1991; Altobelli et al., 1991; Graham et al., 1991; Chow et al., 1993; Mondy et al., 1994; Chow et al., 1994; Koh et al., 1994].

In the case of tube flows, there were controversies over the direction of migration. Nott and Brady [1994] pointed out that previous experimental studies failed to use sufficiently long entrance length to ensure fully developed profiles. Noting this point, Hampton et al. [1997] have carried out very careful experiments on the migration of spherical particles in suspension by using MRI. They have reported that, in the cases of small particle Reynolds number, particles migrate to the low-shear-rate region in the center, and the migration results in the blunting of velocity profile. However, when a/R (particle radius/tube radius) is 0.0656 and the particle loading, ϕ_0, is 0.1, they observed no detectable net radial migration of the particles to the center. They ascribed it to the absence of migration mechanism of Stokes flow. Recently, Han et al. [1999] argued that the absence of migration observed by Hampton et al. could be due to the balance between the inertial effect and particle-particle interaction. They carried out MRI experiments for a wide range of particle loading and particle Reynolds number when a/R=0.12 and found that the inertial effect seemed to play important roles in the tube flow of suspension. Their experimental results are summarized as follows. When ϕ_0, is 0.06, particles are strongly concentrated at the midpoint between the center and the wall. When ϕ_0, is 0.10, the particles move toward the center when Re_p, particle Reynolds number, is vanishingly small. As Re_p becomes larger, particles are almost evenly distributed over the whole region as observed by Hampton et al., and then when Re_p is sufficiently large, particles are strongly concentrated at the midpoint between the center and the wall. The velocity profiles are not distorted from the parabola. When ϕ_0=0.20, compared to the case of ϕ_0=0.10, the transition pattern is qualitatively different. Particles are always found at the center. When Re_p is vanishingly small, particle concentration is almost monotonically decreasing from the center to the wall. When Re_p is large, the concentration profile has a double-humped shape. When ϕ_0=0.40 and Re_p is 0.103, a plateau is observed at the midway between center and tube wall. It should be noted that the experimentally observed plateau or local maximum could not be explained by the Phillips et al.'s model that predicts only monotonic decrease in particle concentration toward the wall.

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‡This paper is dedicated to Professor Hyun-Ku Rhee on the occasion of his retirement from Seoul National University.

Since the particle-particle interaction theory cannot explain the experimental observations for differing Re_p and Re_p is the only relevant parameter in the flow of suspension in Newtonian fluid except a/R, we may predict that the inertial effect has to be taken into account properly in the modeling of suspension flow. In this study, by combining the inertial migration theory by Ho and Leal [1974] and the particle migration model by Phillips et al. [1992], a mathematical model is set up on the particle migration due to particle-particle interaction under the influence of fluid inertia. This model is expected to be valid for the flow of dilute suspensions when the particle Reynolds number is small due to the basic assumptions in the inertial migration theory of Ho and Leal. But the result of model calculation has revealed that the model describes the physical phenomena at least qualitatively even for the case of concentrated suspensions.

FORMULATION OF THE PROBLEM

We consider spherical particles of radius a suspended in an incompressible Newtonian fluid undergoing a plane Poiseuille flow between two parallel and infinite plane walls separated by a distance d as shown in Fig. 1. The particles are assumed to be rigid and neutrally buoyant. Then the particles will migrate due to inertia and particle-particle interaction in the shear field. We assume here that the particles are sufficiently large to neglect the effect of Brownian motion. We also assume that the force exerted on a particle due to inertia is not altered by the presence of other particles. The validity of this assumption will be discussed later in this article.

The mass flux due to inertia between two parallel plates is

$$N_{IM}=\rho\phi U_{IM} \tag{1}$$

where the inertial migration velocity, U_{IM} is given in Ho and Leal as follows:

$$U_{IM}=\frac{F_L^*(z^*)}{6\pi\mu_0 a}=\frac{\rho V_m^{*2} d}{6\pi\mu_0}\kappa^3 G(s) \tag{2}$$

In the above equation, κ is a/d, μ_0 is the viscosity of fluid, V_m^* is the maximum velocity at the center of channel and s is dimensionless coordinate given in Fig. 1. The function G(s) is given in Ho and Leal [1974].

The particle flux due to particle-particle interaction, N_{pp}, is assumed to follow the model by Phillips et al. [1992] According to this model N_{pp} consists of two terms, N_c and N_η, which are due to the concentration gradient and viscosity gradient, respectively, and given as follows:

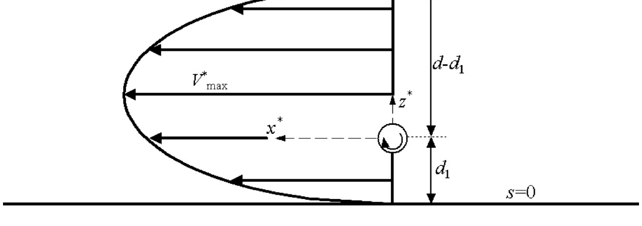

Fig. 1. The geometry of the system for two dimensional Poiseuille flow.

$$N_{pp}=N_c+N_\eta \tag{3}$$

$$N_c=-\rho K_c a^2(\phi^2\nabla\gamma+\phi\gamma\nabla\phi) \tag{4}$$

$$N_\eta=-\rho K_\eta \gamma^2\left(\frac{a^2}{\eta}\right)\frac{d\eta}{d\phi}\nabla\phi \tag{5}$$

In the above equation, γ is shear rate, η is the viscosity of suspension that is a function of particle concentration, and K_c and K_η are constants that should be determined from experiments.

When the flow is fully developed in the steady state, the net flux of particles vanishes. Thus, we finally obtain the governing equation on the particle migration due to inertia and particle-particle interaction.

$$N_{IM}+N_{pp}=0 \tag{6}$$

From the momentum equation

$$\frac{d(\eta\gamma)}{dz^*}=\frac{\Delta p}{L}, \tag{7}$$

shear rate can be determined as a function of position.

$$\frac{d v(z^*)}{dz^*}=\gamma=\frac{\Delta p}{L}\left(z^*-\frac{d}{2}\right)\Big/\eta(\phi). \tag{8}$$

Here γ=0 at the center of the channel. The velocity profile can be obtained by integrating the above equation. The viscosity of suspension can be found by using Krieger's relationship:

$$\mu(\phi)=\frac{\eta(\phi)}{\mu_0}=\left(1-\frac{\phi}{\phi_m}\right)^{-\alpha} \tag{9}$$

To integrate Eqs. (8) and (9) proper boundary conditions are required. In Phillips et al.'s model, the only constraint is that the concentration at the center has the maximum value and the concentration at the wall vanishes. Therefore, the concentration at the center is not known *a priori*. In this study, we use the condition that the average concentration is the same as the particle loading, ϕ_0. We also note that the center of the sphere cannot penetrate into the region adjacent to the wall with the thickness of a. Thus

$$\frac{1}{d}\int_a^{d-a}\phi(z^*)dz^*=\phi_0. \tag{10}$$

We also take into account the discontinuous change of particle concentration in determining the velocity profile. However, since no such problem has been considered theoretically as far as the author is aware, it is assumed that the velocity profile is parabolic as if there exists no particle in the layer. Even though it cannot be rationalized quantitatively, it will be useful in discriminating between blunted and parabolic velocity profiles.

Now we non-dimensionalize the governing equations as follows:

$$\phi\left\{\frac{Re_p}{6\pi}G(s)-K_c\left(\phi\frac{d\beta}{ds}+\beta\frac{d\phi}{ds}\right)-K_\eta\beta\phi\frac{1}{\eta}\frac{d\eta}{d\phi}\frac{d\phi}{ds}\right\}=0 \tag{11}$$

$$\int_\kappa^{1-\kappa}2s\phi(s)ds=\phi_0 \tag{12}$$

$$\beta(s)=\frac{d v(s)}{ds}=-\frac{2}{\mu(\phi)}(2s-1) \tag{13}$$

In the above equation $\beta=\gamma d/V_m^*$ is dimensionless shear rate and s is dimensionless distance from the wall. In integrating the Eq. (13), ν=0 and $\mu(\phi)$=0 should be used at the wall. The set of governing equations can be solved numerically.

Next, we consider flow through a circular tube. For the inertial effect we assume that the same relationship applies as in the case of flow between two parallel plates. This assumption has not been validated yet. However, as pointed out in Ho and Leal [1974], the theoretically predicted equilibrium position of a spherical particle for plane Poiseuille flow is exactly the same as the experimentally observed value in a circular tube by Segre and Silberberg [1962]. Also, the form of the inertial force exerted on a particle in the plane Poiseuille flow is basically the same as Segre and Silberberg's estimate. Therefore, we may use the result for two-dimensional Poiseuille flow in obtaining '*qualitative results*' for the circular Poiseuille flow to examine the role of inertia in particle migration. Under these assumptions, the governing equation can be written in cylindrical coordinates as follows:

$$\phi\left\{\frac{Re_p}{24\pi}G(s) - K_c\left(\phi\frac{d\beta}{ds}+\beta\frac{d\phi}{ds}\right) - K_\eta\beta\phi\frac{1}{\eta}\frac{d\eta}{d\phi}\frac{d\phi}{ds}\right\}=0 \quad (14)$$

$$\int_{\kappa}^{1-\kappa} 2s\phi(s)ds = \phi_0 \quad (15)$$

$$\beta(s) = \frac{d\nu(s)}{ds} = -\frac{2s}{\mu(\phi)} \quad (16)$$

Here R is used as the length scale and s is dimensionless distance from the center.

Han et al. [1999] pointed out that when the particle size is finite and not small, the center of mass distribution determined by solving any migration model is not the same as the solid distribution that is determined by an experimental technique such as MRI. This is especially prominent where the concentration gradient is large such as in wall layers. The relation between the solid distribution, $\bar{\phi}(r)$ and the center of mass distribution, $\phi(r)$ can be found in Han et al. [1999] and given as follows:

$$\bar{\phi}(r) = \int_{-a}^{a}\int_{-\theta_0}^{\theta_0}\psi(p,\theta;r)\,d\theta\,dp \quad (17)$$

where θ_0=arccos[{$(r+p)^2+r^2-a^2$}/2r(r+p)] and

$$\psi(p,\theta;r) = \frac{3}{2\pi a^3}\phi(r+p)\sqrt{a^2-(r+p)^2-r^2+2r(r+p)\cos\theta}\,(r+p).$$

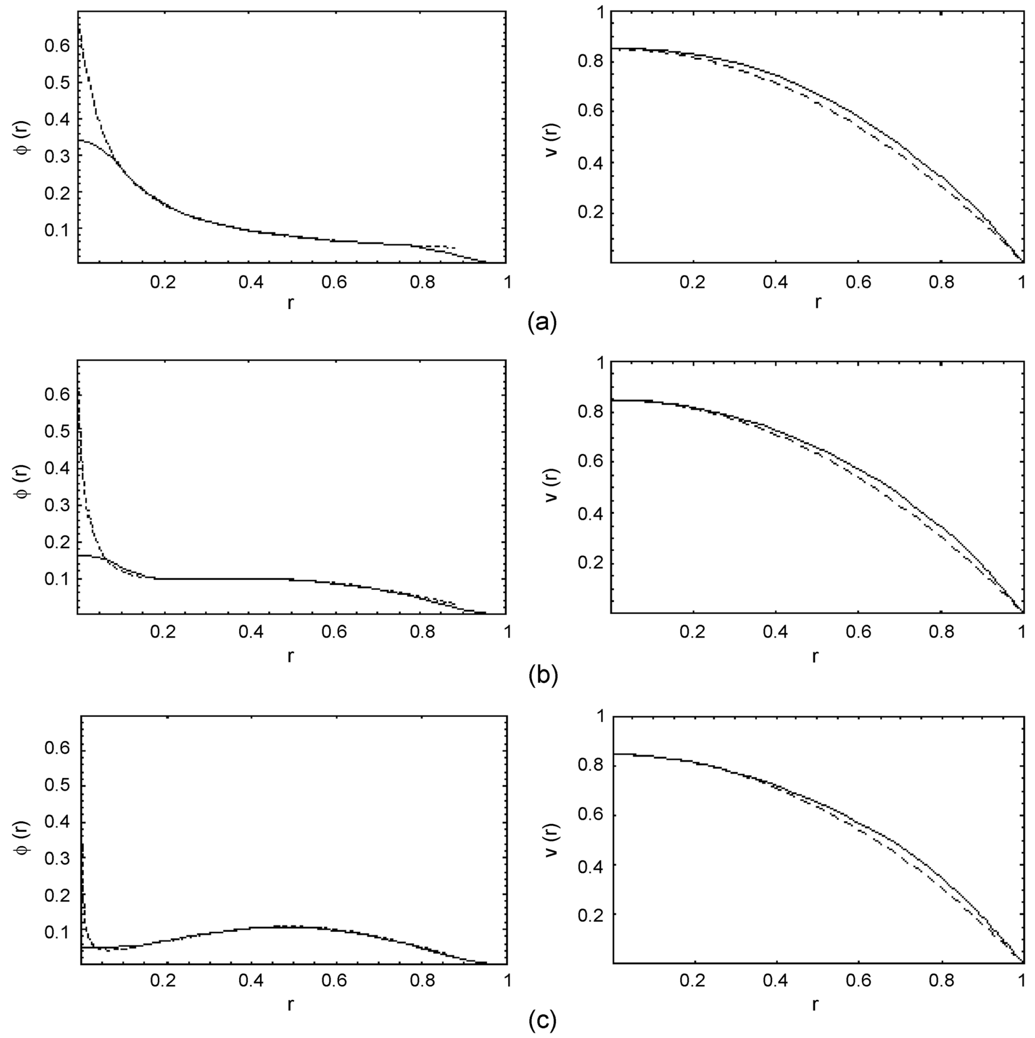

Fig. 2. Particle concentration and velocity profiles when ϕ_0=0.06. The dashed line represents the distribution of center of mass. The solid line is the solid distribution when a/R=0.12.
(a) Re_p=0; (b) Re_p=0.52; (c) Re_p=0.77

When r<a, this above equation should be modified as follows:

$$\bar{\phi}(\mathrm{r}) = \int_{-r}^{a-2r}\int_{-\pi}^{\pi} \psi(\mathrm{p}, \theta; \mathrm{r})\,\mathrm{d}\theta\mathrm{dp} + \int_{a-2r}^{a}\int_{-\theta_0}^{\theta_0} \psi(\mathrm{p}, \theta; \mathrm{r})\,\mathrm{d}\theta\mathrm{dp} \qquad (18)$$

Also, near the wall region, spheres cannot penetrate into the tube wall and Eq. (17) should be modified as follows:

$$\bar{\phi}(\mathrm{r}) = \int_{-a}^{a^*}\int_{-\theta_0}^{\theta_0} \psi(\mathrm{p}, \theta; \mathrm{r})\,\mathrm{d}\theta\mathrm{dp} \qquad (19)$$

Where a^* is dependent on r: $a^*=a$ if $a \le r \le R-2a$; $a^*=R-a-r$ if $R-2a \le r \le R-a$; $a^*=r-R+a$ if $R-a \le r \le R$.

Now we can determine particle concentration and velocity profiles numerically. In the next section, we describe the results of model calculations.

RESULTS OF MODEL CALCULATION

Numerical solutions were obtained for the case of the circular Poiseuille flow for a range of Reynolds numbers for selected ϕ_0 values by using Mathematica™ Software. The numerical values of K_c and K_η were adopted from Phillips et al. [1992] (K_c=0.41, K_η=62).

Fig. 2 shows the radial profiles of particle concentration and velocity when ϕ_0=0.06. In this case, we can notice the transition from the dominant hydrodynamic-interaction when Re_p=0 to the dominant inertia when Re_p is larger than 0.7. When Re_p is zero, the particles move toward the center. As reviewed in the previous section, Han et al. performed experiments only when the inertial effect was dominant. It was because, in the limit of vanishingly small Reynolds number, particles rose or sank due to slight density-mismatch that was unavoidable. Hence we cannot compare the model calculation with experimental results. It is predicted that a single particle will not migrate in Stokes flow [Happel and Brenner, 1983]. But when the particle concentration is not vanishingly small, particles will migrate due to particle-particle interaction. Since there is no inertial effect, particles will move toward the center where the shear rate has the minimum value and hence the energy dissipation is the smallest. When Re_p is not zero, particles move toward the

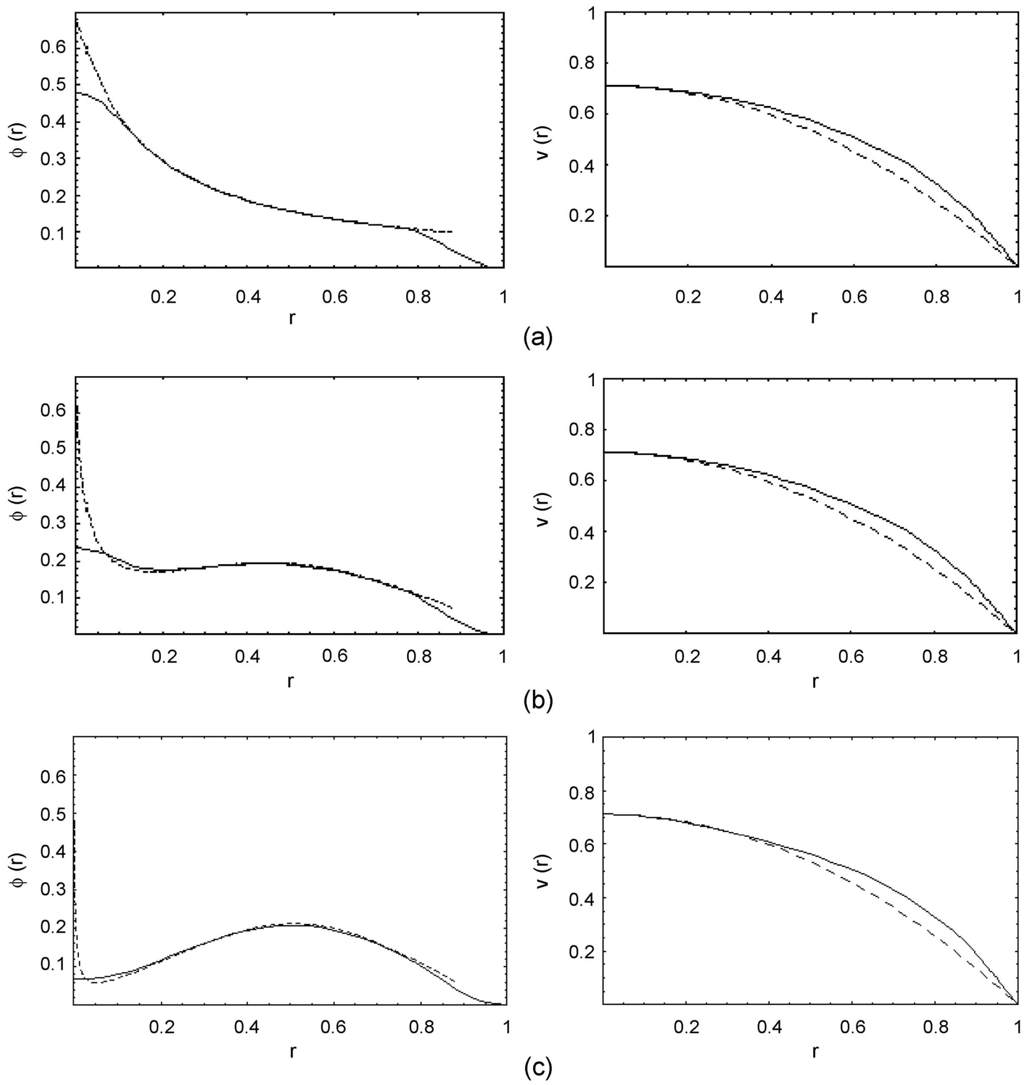

Fig. 3. Particle concentration and velocity profiles when ϕ_0=0.12. The dashed line represents the distribution of center of mass. The solid line is the solid distribution when a/R=0.12.
(a) Re_p=0; (b) Re_p=0.76; (c) Re_p=1.13

midway between the wall and the tube axis under the influence of inertia as in the case of a single particle. But as many particles are gathered in a specific region, particle-particle interaction should become effective. In the limit of dominant inertial effect, the model predicts that particles are found near the tube axis as well as at the midway between the wall and tube axis even though the particles the centers of which are located near the axis do not contribute much to the solid concentration near the tube axis. However, particle concentration near the tube axis is virtually zero in the experiments [Han et al., 1999; Segre and Silberberg, 1962]. This appears to be caused by the singularity at the tube axis in the Phillips et al.'s model. In other words, Phillips et al.'s model predicts that, at the tube axis, the particle concentration has to be the maximum packing fraction. However, when we consider the solid concentration rather than the center of mass distribution as explained by Eq. (17), the model predicts that particles are mostly found at the midway between the wall and tube axis as reported by Han et al. (Dashed line in Fig. 2). Next we note that the location of the maximum concentration is shifted from r/R=0.6 to 0.5. The shift has also been reported in the experimental work of Han et al. This shift appears to be caused by the particle-particle interaction. As a summary for the very dilute case, the model calculation explains the experimental results from MRI for large particle Reynolds number. But when Re_p is small, the particle-particle interaction term from the Phillips model appears to be too much exaggerated. It is not clear whether this difference is caused by the assumptions we introduced or the singular behavior of Phillips et al.'s model at the tube axis. Also, we may obtain a better fit by changing the numerical parameters in the model because the values used here were determined from the concentrated suspension and they may not be valid for this low value of particle loading. Since the purpose of this model calculation was to investigate the qualitative effect of inertia, we have not performed any further parametric studies by changing the numerical values. In the case of velocity profiles, the parabolic form is maintained, which is also confirmed by the MRI experiment.

Fig. 3 shows the model calculation result when ϕ_0=0.12. When Re_p is 0, particles concentrate at the center of tube as in the case of ϕ_0=0.06. When Re_p is 0.76, particles are distributed almost evenly throughout the cross section except near the wall where the no penetration condition prohibits the existence of a center of particle with

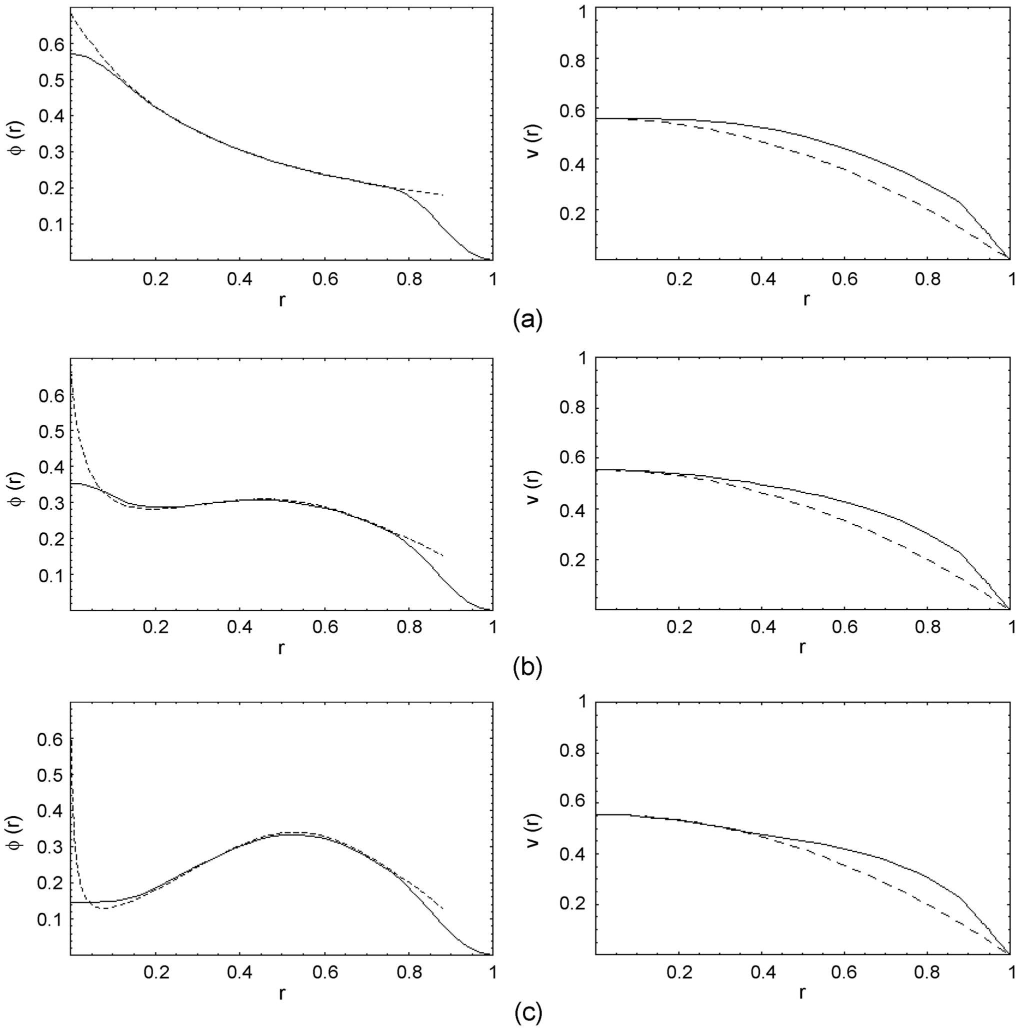

Fig. 4. Particle concentration and velocity profiles when ϕ_0=0.20. The dashed line represents the distribution of center of mass. The solid line is the solid distribution when a/R=0.12.
(a) Re_p=0; (b) Re_p=0.64; (c) Re_p=1.13

finite size. When Re_p is sufficiently large, particles are concentrated midway between the wall and the tube axis. The qualitative pictures of transition from low Re_p to high Re_p exactly matched the experimental observation by Han et al. [1999] as described in the introduction. However, compared to the experimental results by MRI, the inertia seems to be too strong. This is due to the fact that we used the inertial migration model that was developed for a single particle. Hence, in the particle loading considered here, the inertial effect appears to be overestimated. This is possible considering that the rotational motion of a particle is hindered by the presence of other particles. When Re_p is 0.76, the concentration profile has a slightly double-humped shape. The maximum at the tube axis originates from the strong particle-particle interaction and the local maximum at r=0.5 is from the inertial effect. The double-humped profile was not observed in Han et al.'s experiment when ϕ_0=0.1. The resolving power of their MRI was not strong enough to differentiate such small difference in particle concentration. In the velocity profile, we notice an inflection point near r=0.4 and flatted profile near r=0.5 when Re_p=1.13. This is caused by the large particle concentration near r=0.5. But the experiment could not catch such a small difference even though such a difference exists in real systems. It seems such a small signal was attenuated during the averaging process of signal processing.

Fig. 4 shows the model calculation result when ϕ_0=0.20. Since the particle loading is too high, the inertial effect should be extremely overestimated. However, it may be useful to investigate the inertial effect qualitatively. When Re_p is 0, particle concentration decreases monotonically from the maximum packing fraction. When Re_p is 0.64, the particle concentration profile has double humps. When Re_p is sufficiently large, particles are concentrated midway between the wall and the tube axis. In this case, the inertial effect is again too much exaggerated. However, the double hump is also observed experimentally when ϕ_0=0.20; hence we can confirm that the experimentally observed phenomena are the manifestation of inertial effects and the particle-particle interaction. Velocity profiles become blunted slightly as the particle loading increases, which is also observed in the MRI experiment.

SUMMARY

In this research, we have considered the migration of particles in a tube flow of suspension by setting up a model by combining the inertial migration theory and the particle-particle interaction model. The model set up here explains the experimental observation when Re_p<1 at least qualitatively. We have found that a fluid's inertia may not be neglected even for the flow of concentrated suspensions when Re_p is larger than approximately 0.1. Also, once particles are concentrated at a certain position, particle-particle interaction tends to spread them out. This was also applied to the cases with low ϕ_0. Therefore, in the migration of particles in suspension, inertia as well as particle-particle interaction should be taken properly into account regardless of particle loading.

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